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Ivo Düntsch & Günther Gediga

Sets, Relations, Functions

Methoδos

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Sets, Relations, Functions

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Chapter 1

Sets

1.1 Introduction to Sets

Mathematics deals with objects of very different kinds; from your previous experience, you are familiar with many of them: Numbers, points, lines, planes, triangles, circles, angles, equations, functions and many more. Often, objects of a similar nature or with a common property are collected into sets; these may be finite or infinite (For the moment, it is enough if you have an intuitive understanding of finite, resp. infinite; a more rigorous definition will be given at a later stage). The objects which are collected in a set are called the *elements* of that set. If an object a is an element of a set M, we write

$$a \in M$$

which is read as *a (is an) element of M*.

If a is not an element of M, then we write

$$a \notin M$$

which is read as *a is not an element of M*.

Example 1.1.1.

1. If Q is the set of all quadrangles, and A is a parallelogram, then $A \in Q$. If C is a circle, then $C \notin Q$.
2. If G is the set of all even numbers, then $16 \in G$, and $3 \notin G$.
3. If L is the set of all solutions of the equation $x^2 = 1$, then 1 is an element of L, while 2 is not.

Generally, there are two ways to describe a set:

- By listing its elements between curly brackets and separating them by commas, e.g.

$$\{0\},$$
$$\{2, 67, 9\},$$
$$\{x, y, z\}.$$

  Note that this is convenient (or indeed possible) only for sets with relatively few elements. If there are more elements and one wants to list the elements "explicitly" sometimes periods are used; for example,

$$\{0, 1, 2, 3, \ldots\}, \{2, 4, 6, \ldots, 20\}.$$

  The meaning should be clear from the context. In this descriptive or explicit method, an element may be listed more than once, and the order in which the elements appear is irrelevant. Thus, the following all describe the same set:

$$\{1, 2, 3\},\ \{2, 3, 1\},\ \{1, 1, 3, 2, 3\}.$$

- By giving a rule which determines if a given object is in the set or not; this is also called implicit description; for example,

  1. $\{x : x$ is a natural number$\}$
  2. $\{x : x$ is a natural number and $x > 0\}$
  3. $\{y : y$ solves $(y + 1) \cdot (y - 3) = 0\}$
  4. $\{p : p$ is an even prime number$\}$.

  Usually, there is more than one way of describing a set. Thus, we could have written

  1. $\{0, 1, 2, ...\}$
  2. $\{1, 2, 3, ...\}$
  3. $\{-1, 3\}$
  4. $\{2\}$ or $\{x : 2x = 4\}$

The general situation can be described as follows: A set is determined by a defining property P of its elements, written as

$$\{x : P(x)\}$$

where $P(x)$ means that x has the property described by P. The letter x serves as a variable for objects; any other letter, or symbol except P, would have done equally well; similarly, P is a variable for properties or, as they are sometimes called, *predicates*. To avoid running into logical difficulties we shall always assume that our objects which are described by the predicate P come from a previously well defined set, say M, and sometimes we shall say so explicitly. In general then, we describe sets by

$$\{x : x \in M \text{ and } P(x)\},$$

which also can be written as

$$\{x \in M : P(x)\}.$$

We shall use the following conventions in describing certain sets of numbers:

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of *natural numbers*.
- $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ is the set of *positive natural numbers*.
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of *integers*.
- $\mathbb{Q} = \{x : x = \frac{a}{b}\}$, where $a \in \mathbb{Z}, b \in Z$, and $b \neq 0$, is the set of *rational numbers*; observe that each rational number is the ratio of two integers, whence the name.
- $\mathbb{R} = \{x : x \text{ is a real number}\}$.

We shall not give a rigorous definition of a real number; it is assumed that you have an intuitive idea of the reals - think of them as being the points on a straight line.

1.1.1 Exercises

Exercise 1. Describe the following sets explicitly:

$$\begin{aligned}
A &= \{x \in \mathbb{Z} : x \lneq 5\} \\
B &= \{x \in \mathbb{N} : x \text{ divides } 24\} \\
C &= \{x \in \mathbb{R} : x = 2\} \\
D &= \{x \in \mathbb{N}^+ : x \lneq 0\}
\end{aligned}$$

Exercise 2. Find a property P and a set M such that you can write the following sets in the form $\{x \in M : P(x)\}$:

$$A = \{1, 2, 4, 8, 16\}$$
$$B = \{6, 4, 8, 2, 0\}$$
$$C = \{1, 3, 5, 7, ...\}$$
$$D = \{3, 5, 11, 2, 7, 13\}$$

1.2 Subsets, Power Sets, Equality of Sets

Definition 1.2.1. A set A is a *subset of a set B*, written as $A \subseteq B$, if every element of A is also an element of B. The relation $\subseteq$ is called the *inclusion relation*. □

So, $A \subseteq B$ whenever $x \in A$ implies $x \in B$. Observe carefully the difference between $\subseteq$ and $\in$:

If $B = \{1, 2, 3\}$, then 1 is an element of B, but 1 is not a subset of B. The set $A = \{1\}$, which has 1 as its only element, however, is a subset of B, since it fulfils the definition: Whenever $x \in A$, then $x \in B$. Note that there is only one possibility for x, namely, $x = 1$. It is very important that you learn to distinguish between an object (which may of course be a set itself), and the set formed from objects, e.g.

- 1 is different from $\{1\}$,
- $\{1\}$ is different from $\{\{1\}\}$.

Observe that the unique element of the last set is the set $\{1\}$.

It is conceivable that a set contains no elements at all; this set is called the empty set, and it is denoted by the symbol $\emptyset$. It can also be described by a property, e.g.

$$\emptyset = \{x : x \neq x\}.$$

Here are some elementary properties of the $\subseteq$ relation:

Lemma 1.2.1. *$\emptyset$ is a subset of every set.*

Proof. Recall that by definition, $\emptyset$ is a subset of a set A, if every element of $\emptyset$ is also an element of A. Since $\emptyset$ has no elements, this is trivially true. □

Lemma 1.2.2. *For any set A, $A \subseteq A$.*

Proof. By our definition of $\subseteq$, A is a subset of A, if every element of A is an element of A, and this is of course true. $\square$

Lemma 1.2.3. *If A is a subset of B, and B is a subset of C, then A is a subset of C.*

Proof. We want to show that $A \subseteq C$, and we have as hypothesis that $A \subseteq B$ and $B \subseteq C$. Using our definition of $\subseteq$, we have to show that if $x \in A$, then x is an element of C. Thus, let x be an arbitrary element of A. Our hypothesis $A \subseteq B$ tells us that $x \in B$, and, since $B \subseteq C$, we also have $x \in C$. This is what we wanted to prove. $\square$

The next definition allows us to form a new set from a given one:

Definition 1.2.2. If A is a set, then $\mathfrak{P}(A) = \{X : X \subseteq A\}$ is called the *power set of A*. It is the set of all subsets of A. $\square$

Let us look at some power sets:

1. $A = \emptyset$: Since $\emptyset$ is a subset of every set, we must have $\emptyset \subseteq \emptyset$, i.e. $\emptyset \in \mathfrak{P}(\emptyset)$. But $\emptyset$ has no elements, and therefore no other subsets; hence, we have $\mathfrak{P}(\emptyset) = \{\emptyset\}$. Observe that $\emptyset$ is different from $\{\emptyset\}$: While $\emptyset$ has no elements, $\{\emptyset\}$ has exactly one element, namely the empty set $\emptyset$.

2. $A = \{x\}$: By Lemma 1.2.1 and Lemma 1.2.2, A has at least two subsets, namely the empty set $\emptyset$, and A itself. Since A has only one element, there can be no other subsets,thus, $P(A) = \{\emptyset, \{x\}\}$.

3. $A = \{x, y\}$: Using the previous lines of reasoning we obtain that

$$\mathfrak{P}(A) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}. \tag{1.1}$$

   So, we see that $\mathfrak{P}(A)$ has four elements.

4. $A = \{x, y, z\}$: Except for the subsets $\emptyset$ and A, we can pick one or two elements of A at a time to form a subset. Hence,

$$\mathfrak{P}(A) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}.$$

   Observe that $\mathfrak{P}(A)$ has eight elements.

When shall we call two sets equal? An intuitive definition is to say that two sets are equal if they contain the same elements. We shall use our already defined notion $\subseteq$ to define equality of sets:

Definition 1.2.3. Two sets A and B are equal, if $A \subseteq B$ and $B \subseteq A$. If A and B are equal, we write $A = B$. $\square$

Observe that this definition expresses just what we would intuitively mean by equality of sets: If every element of A is an element of B,and if every element of B is an element of A, then they must contain the same elements. To prove that two sets A and B are equal, we must show that A is a subset of B, and that B is a subset of A. The procedure shall be illustrated by an easy

Example 1.2.1. Let $A = \{x \in \mathbb{R} : x^2 = 1\}$, and $B = \{1, -1\}$; we want to show that $A = B$:

"$A \subseteq B$": Let $x \in A$; then, by the definition of A, x solves the equation $x^2 = 1$, hence, $x = 1$ or $x = -1$. In either case, $x \in B$.

"$B \subseteq A$": Let $x \in B$; then, by definition of B, $x = 1$ or $x = -1$. In either case, x is a real number and solves the equation $x^2 = 1$, hence, it fulfils the defining properties of A. This implies that $x \in A$.

1.2.1 Exercises

Exercise 3. Find the power set of $\{a, b, c, d\}$; how may elements does it have?

Exercise 4. Using the examples of power sets above and your answer to the problem above, conjecture the answer to the following problem:

> Given that a set A has n elements, where n is a fixed natural number, how many elements does $\mathfrak{P}(A)$ have? You need not prove your answer, but explain your answer.

Exercise 5. Prove:

If $A = B$ and $B = C$, then $A = C$.

Prove this Lemma for the general case; an example is not a proof!

1.3 Finite and Infinite Sets

Definition 1.3.1. A set M is called *finite*, if $M = \emptyset$, or if there is natural number n such that the elements of M can be numbered $1, \ldots, n$ in such a way that every element of M appears exactly once in the list. Otherwise, M is called *infinite*. □

Example 1.3.1.

1. The set $\{a, b, c, d\}$ is finite. A possible numbering might look like this: Assign 1 to a, assign 2 to b, assign 3 to c, assign 4 to d. Of course there are other possibilities of listing the set M (Which?).
2. The set of all solutions of the equation $x^2 + 23x - 17 = 0$ is finite, since the number of solutions of a polynomial is at most equal to the degree.
3. The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are infinite.
4. The set of all multiples of 5 is infinite.

Sometimes, it is not immediately clear, if a set is finite or infinite. One of the famous problems in Mathematics was the question for which natural numbers n there exist positive natural numbers a, b, c such that $a^n + b^n = c^n$. This question is called *Fermat's last problem*. If $n = 2$, then we obtain the Pythagoraic equation, and e.g. $a = 3, b = 4, c = 5$ is a solution. It was only shown recently by A. Wiley that 2 is in fact the only integer for which such numbers exist.

Another problem concerns twins of primes. As you will recall, a prime number p is a natural number greater than 1 which is divisible only by 1 and by itself. A pair of prime numbers $\langle p, q\rangle$ is called a *twin pair*, if $p + 2 = q$, i.e. they are consecutive odd numbers. Examples of twin pairs are $\langle 5, 7\rangle$, $\langle 11, 13\rangle$, $\langle 59, 61\rangle$. It is not known, whether there are infinitely many prime pairs.

The following theorem has been known since the time of Euclid (ca. 300 BC):

Theorem 1.3.1. *There are infinitely many prime numbers.*

Proof. Following Euclid's proof, we shall show that to every prime p there is a greater one. Assume that p is the greatest prime number, and let $q = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot p$. Then, $q + 1$ is not divisible by $2, 3, \ldots, p$. It follows that q is divisible only by 1 and itself, and thus, it is a prime number greater than p. This, however, contradicts the hypothesis that p is the greatest prime, and it follows that there is no greatest prime. In other words, the set of primes is infinite. □

1.3.1 Exercises

Exercise 6.

1. Explain why $\mathbb{N}$ is infinite.
2. Explain why $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are infinite.

Exercise 7. Are the following sets finite or infinite?

1. $\{x \in \mathbb{R} : x^2 + 2x - 1 = 0\}$
2. $\{x \in \mathbb{N} : x \lneq 0\}$
3. $\{x \in \mathbb{Q} : 0 \lneq x \lneq 1\}$

1.4 Set Operations

In this section we shall encounter several ways of obtaining sets from previously given ones.

Definition 1.4.1. Let A and B be given sets; then

1. The *intersection* $A \cap B$ of A and B is defined by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

   If $A \cap B = \emptyset$, then A and B are called *disjoint*.

2. The *union* $A \cup B$ of A and B is defined by

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

3. The *set difference* $A \setminus B$ of A and B is the set

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

   $A \setminus B$ is also called the *relative complement of* B *with respect to* A.

   If U is a given universal set, then $U \setminus A$ is just called the *complement of* A, written as $-A$. □

Observe that $A \cap B$ is the set of those objects which are simultaneously in both A and B, while $A \cup B$ is the set of those objects which are in A or in B or in both of them; observe carefully that we do not interpret "or" as exclusive; in our terminology, "or" always means "one, or the other, or both".

Example 1.4.1. 1. Let

$$A = \{x \in \mathbb{Z} : -1 \leq x \lneqq 4\} = \{-1, 0, 1, 2, 3\},$$
$$B = \{x \in \mathbb{Z} : 1 \lneqq \frac{x}{2} \lneqq 7\} = \{2, 3, 4, 5, 6\}.$$

Then,

$$\begin{aligned} A \cap B &= \{x \in \mathbb{Z} : x \in A \text{ and } x \in B\} \\ &= \{x \in \mathbb{Z} : -1 \lneqq x \lneqq 4 \text{ and } 1 \lneqq \frac{x}{2} \lneqq 7\} \\ &= \{2, 3\}. \end{aligned}$$

Furthermore,

$$\begin{aligned} A \cup B &= \{x : x \in A \text{ or } x \in B\} \\ &= \{x \in \mathbb{Z} : -1 \leq x \lneqq 4 \text{ or } 1 \lneqq \frac{x}{2} \lneqq 7\} \\ &= \{-1, 0, 1, 2, 3, 4, 5, 6\} \end{aligned}$$

2. Let g_1 and g_2 be two non–parallel lines in the plane. Then, their intersection $g_1 \cap g_2$ is just the point where the two lines meet. Their union $g_1 \cup g_2$ is the set of all points which are on g_1 or on g_2 (or on both lines).

3. Let R be the set of all station wagons in Iceland at a given time, and let S be the set of all white cars in Iceland. Then $S \cap T$ is the set of all white station wagons in Iceland, and $S \cup T$ is the set of all those cars in Iceland which are a station wagon or which are white or both.

4. Let $U = \mathbb{N}$, and $A = \{x \in \mathbb{N} : x \text{ is prime}\}$. Then $-A$ is the set of all those natural numbers which are less than 2 or have at least three divisors.

5. Let $U = \mathbb{N}$, and A be the set of all even natural numbers. Then $-A$ is the set of all odd natural numbers.

A convenient pictorial representation of the operations defined above are the *Venn diagrams*, shown in Fig. 1.1 – 1.3 on the following page: The shaded area represents the result of the respective operation.

Lemma 1.4.1. *Let $A \subseteq U$; then*

1. $A \cap A = A, \ A \cup A = A.$

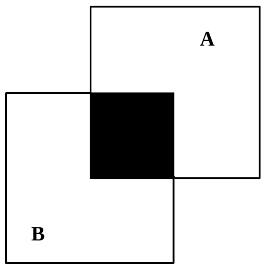


Figure 1.1: $A \cap B$

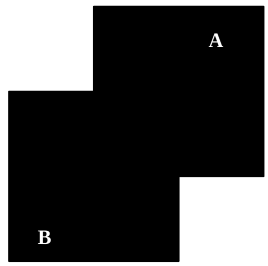


Figure 1.2: $A \cup B$

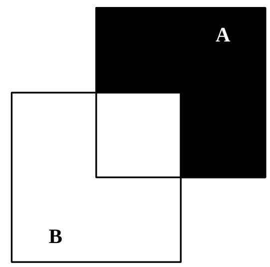


Figure 1.3: $A \setminus B$

2. $A \cap \emptyset = \emptyset,\ A \cup \emptyset = A$.

3. $A \cap U = A,\ A \cup U = U$.

4. $A \cap -A = \emptyset,\ A \cup -A = U$.

Proof. We shall only show 1. and 2.; the rest is left as an exercise.

1. $A \cap A = \{x \in U : x \in A$ and $x \in A\} = \{x \in U : x \in A\} = A$.

$A \cup A = \{x : x \in A$ or $x \in A\} = \{x : x \in A\} = A$.

2. $A \cap \emptyset = \{x : x \in A$ and $x \in \emptyset\}$; since $\emptyset$ has no elements, the right hand side (and thus the left hand side) of the equation is the empty set.

$A \cup \emptyset = \{x : x \in A$ or $x \in \emptyset\}$; again, since $\emptyset$ has no elements, we do not "add" anything to A, hence, $A \cup \emptyset = A$. □

Lemma 1.4.2. *If A and B are sets, then $A \cap B = B \cap A$, and $A \cup B = B \cup A$. This is called the* Law of Commutativity for $\cap$ and $\cup$.

Proof. We only prove the first part; the second part is similar:

$$A \cap B = \{x : x \in A \text{ and } x \in B\} = \{x : x \in B \text{ and } x \in A\} = B \cap A.$$

□

Since we have defined the operations $\cap$ and $\cup$ only for two sets, the expressions $A \cap B \cap C$ and $A \cup B \cup C$ do not make sense. However, if we write $(A \cap B) \cap C$ and $(A \cup B) \cup C$, then this instruct us first to find the intersection (union) of A and

B, and then the intersection (union) of $A \cap B$ ($A \cup B$) with C. On the other hand, we could also have interpreted the line $A \cap (B \cap C)$ and $A \cup (B \cup C)$, which tells us first to find the intersection (union) of B and C, and then proceed to A. The following Lemma shows that it does not matter what we do first - the result is the same.

Lemma 1.4.3. *If A, B, C are sets, then*

1. $(A \cap B) \cap C = A \cap (B \cap C)$
2. $(A \cup B) \cup C = A \cup (B \cup C)$

This is called the Law of Associativity *for $\cap$ and $\cup$.*

Proof. We only prove 1. We have to show both inclusions, i.e If $x \in (A \cap B) \cap C$, then $x \in A \cap (B \cap C)$ and vice versa. "$\subseteq$": Let $x \in (A \cap B) \cap C$; then, $x \in A \cap B$ and $x \in C$; since x is an element of $A \cap B$, we obtain that $x \in A$ and $x \in B$. Thus we have $x \in A$ and $x \in B$ and $x \in C$, which implies $x \in A \cap (B \cap C)$.

The proof of the other inclusion is left as an exercise. □

The previous Lemma enables us to give a meaningful interpretation to expressions like $A \cap B \cap C$ or $A \cup B \cup C$, e.g $A \cup B \cup C = \{x : x \in A \text{ or } x \in B \text{ or } x \in C\}$, and we shall omit the brackets where no confusion can arise. The law of associativity also allows us to consider more than three sets by grouping the sets in a convenient way.

1.5 De Morgan Rules, Distributivity, Tables

In the sequel, let U be a universal set, of which all other mentioned sets are subsets. Our first Theorem in this section exhibits a connection between intersection, union, and complementation:

Theorem 1.5.1. *For all sets A, B*

$$-(A \cup B) = -A \cup -B, \; -(A \cap B) = -A \cup -B$$

These are the Rules of De Morgan*.*

Proof. We only show the first part and leave the second as an exercise. "$\subseteq$": Let $x \in -(A \cup B)$; then x is not in the union of A and B, hence, x is neither in A nor in B. Since x is not in A, we have $x \in -A$, and since x is not in B, we have $x \in -B$. Thus, $x \in -A$ and $x \in -B$ which implies that $x \in -A \cap -B$.

"$\supseteq''$: Conversely, let $x \in -A \cap -B$; then x is not in A and x is not in B. Hence, x is not an element of A or B, i.e. $x \in -(A \cup B)$. $\square$

Even though the proofs of set equations are simple, they can be quite tedious, and we shall introduce a new tool for tackling problems of this sort.

If A and B are sets, then for an arbitrary element x of our universe of discourse U there are four possibilities:

$$x \in A \text{ and } x \in B,\ x \in A \text{ and } x \notin B\ x \notin A \text{ and } x \in B,\ x \notin A \text{ and } x \notin B.$$

For every one of these cases let us consider if x is in the intersection of A and B: If $x \in A$ and $x \in B$, then $x \in A \cap B$. In all three other cases, we have $x \notin A \cap B$. This observation can be put in the form of a table which looks like this:

A	B	$A \cap B$	$A \cup B$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	0

Here, an entry 1 means that x is in the set pointed to by the column head, and 0 tells us that it is not.

Regarding complementation, we have the following table:

A	-A
0	1
1	0

With these three tables we can find the table for every set which is formed from A and B by the operations $\cap$, $\cup$, and $-$. Let us look at the table for the first De Morgan rule:

A	B	$-A$	$-B$	$A \cup B$	$-(A \cup B)$	$-A \cap -B$
1	1	0	0	1	0	0
1	0	0	1	1	0	0
0	1	1	0	1	0	0
0	0	1	1	0	1	1

Observe that the right columns of the table have the same entries of in the corresponding places. This tells us that the two sets are equal.

Theorem 1.5.2. *For all sets A, B*

$$A \cup (A \cap B) = A, \; A \cap (A \cup B) = A.$$

These are the absorption laws.

Proof. We shall only prove the first equation, and use a table:

A	B	$A \cap B$	$A \cup (A \cap B)$
1	1	1	1
1	0	0	1
0	1	0	0
0	0	0	0

This proves our claim. □

The method of tables can also be used with more than just two sets; then, of course, the number of possibilities increases, e.g. if we have sets A, B, C, then there are eight possibilities for an arbitrary $x \in U$, and if we have A, B, C, D then there are already sixteen. In general, if there are n sets, then a column in a table for each of these n sets will have 2^n entries.

Our final Theorem in this section reveals the interplay between $\cap$ and $\cup$:

Theorem 1.5.3. *For all sets A, B, C,*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \; A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

These are the distributive Laws for $\cap$ and $\cup$.

Proof. Exercise. □

1.5.1 Exercises

Exercise 8. Using the method of tables, prove

1. The second De Morgan rule,
2. The second Absorption Law,
3. The first Distributive Law.

Chapter 2

Relations

2.1 Ordered Pairs, Cartesian Products

As stated previously, if a set A is given explicitly, it is immaterial in which order the elements of A are listed, e.g. the set $\{x, y\}$ is the same as the set $\{y, x\}$. In many instances, however, one would like, and, indeed, needs, to have some order in the appearance of the elements.

As an example, consider a coordinate plane with an x-axis and a y-axis; then we can identify any point in the plane by its coordinates $\langle x, y\rangle$. If you wanted to find the point $\langle a, b\rangle$, you would move on the x-axis a units to the right or to the left from the origin (depending on the sign of a), and then you would move b units up or down. If a and b are different, then $\langle a, b\rangle$ and $\langle b, a\rangle$ denote different points. So, in this example the order in which the elements appear is relevant. This leads to the following

Definition 2.1.1. Let A be a set;

1. A is called a *singleton* if $A = \{x\}$ for some x, i.e. if A has exactly one element.

2. A is called an *unordered pair*, if $A = \{x, y\}$ for some x, y, if A has exactly two elements.

3. A is called an *ordered pair* if $A = \{\{x\}, \{x, y\}\}$ for some x, y. $\square$

We shall usually abbreviate the right hand expression by

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}.$$

The decisive property of ordered pairs is that two ordered pairs are equal if the respective components are the same. The following Theorem assures us, that the ordered pair as we have defined it has this property:

Theorem 2.1.1. *Let $\langle a, b\rangle$ and (c, d) be ordered pairs. Then$\langle a, b\rangle = \langle c, d\rangle$ if and only if $a = c$ and $b = d$.*

Remark. The expression "if and only if" means that

1. If $\langle a, b\rangle = \langle c, d\rangle$, then $a = c$ and $b =$ d.
2. If $a = c$ and $b = d$, then $\langle a, b\rangle = \langle c, d\rangle$. (This is called "the converse" of 1.)

So, we have to prove two directions, namely 1. and 2. Usually, "if and only if" is abbreviated as simply "iff". □

Proof. "⇒": Suppose $\langle a, b\rangle = \langle c, d\rangle$; then, by definition,

$$\langle a, b\rangle = \{\{a\}, \{a, b\}\},$$
$$\langle c, d\rangle = \{\{c\}, \{c, d\}\},$$

and since they are equal by our hypothesis, we have

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}.$$

We consider two cases:

1. $a = b$: Then,

$$\langle a, b\rangle = \{\{a\}, \{a, b\}\} = \{\{a\}\} = \{\{c\}, \{c, d\}\},$$

hence, $\{a\} = \{c\}$ which implies $a = c$. Furthermore, $\{a\} = \{c, d\} = \{a, d\}$ which implies $d = a = b$. Thus, for this case we have shown that $a = c$ and $b = d$.

2. $a \neq b$: Then, $\{a\} \neq \{a, b\}$. By our hypothesis we must have $\{a\} = \{c\}$, since both sets have only one element; this implies $a = c$.

Furthermore, $\{a, b\} = \{c, d\}$, since $\{a, b\}$ has two elements. We have just shown that $a = c$, thus, $\{a, b\} = \{c, b\} = \{c, d\}$. This implies $b = d$, and this is what we wanted to show.

"$\Leftarrow$": For the converse, let $a = c$ and $b = d$; then, $\{a\} = \{c\}$, and $\{a, b\} = \{c, d\}$, hence,

$$\langle a, b\rangle = \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} = \langle c, d\rangle,$$

which is what we wanted to show. $\square$

You need only remember the property of ordered pairs given in the preceding Theorem; it is not necessary to remember the set theoretic definition.

Definition 2.1.2. The *Cartesian (or cross-) product $A \times B$ of two sets* is defined as

$$A \times B = \{\langle a, b\rangle : a \in A,\ b \in B\}$$

So, the operation $\times$ pairs the elements of A with the elements of B in such a way that the elements of A appear as first components, and the elements of B appear as second components. It is also possible to define Cartesian products for more than two factors, but we shall not do this at this stage.

Example 2.1.1.

1. Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$; then

$$A \times B = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 2, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle, \langle 3, b\rangle.$$

2. Let $A = \mathbb{R}$ and $B = \mathbb{R}$; then $A \times B = \mathbb{R} \times \mathbb{R}$ is the set of all points in the Cartesian coordinate plane.

3. Let $A = B = \{x \in \mathbb{R} : -1 \lneq x \lneq 1\}$; then $A \times B$ is the set of all points in the plane which lie inside a square of side length 2 with centre at the origin.

2.1.1 Exercises

Exercise 9. Let $A = \{2, 4, 6\}$ and $B = \{4, 8, 12\}$; find $A \times B$ and $B \times A$.

Exercise 10. Is it possible that $A \times B = B \times A$? Explain your answer.

Exercise 11. If A has n elements and B has m elements, how many elements does $A \times B$ have? Explain your answer.

2.2 Introduction to Relations

Sometimes it is necessary not to look at the full Cartesian product of two sets A and B, but rather at a subset of the Cartesian product. This leads to the following

Definition 2.2.1. Any subset of $A \times B$ is called a *relation between A and B*.

Any subset of $A \times A$ is called a *relation on A*. □

In other words, if A is a set, any set of ordered pairs with components in A is a relation on A. Since a relation R on A is a subset of $A \times A$, it is an element of the powerset of $A \times A$, i.e. $R \subseteq \mathfrak{P}(A \times A)$. If R is a relation on A and $\langle x, y \rangle \in R$, then we also write xRy, read as "x is in R-relation to y", or simply, *x is in relation to y*, if R is understood.

Example 2.2.1.

1. Let $A = \{2, 4, 6, 8\}$, and define the relation R on A by $\langle x, y \rangle \in R$ iff x divides y. Then,

   $$R = \{\langle 2, 2\rangle, \langle 2, 4\rangle, \langle 2, 6\rangle, \langle 2, 8\rangle, \langle 4, 4\rangle, \langle 4, 8\rangle, \langle 6, 6\rangle, \langle 8, 8\rangle\}.$$

   Observe that each number is a divisor of itself.

2. Let $A = \mathbb{N}$, and define $R \subseteq A \times A$ by

   xRy iff x and y have the same remainder when divided by 3.

   Since A is infinite, we cannot explicitly list all elements of R; but, for example

   $$\langle 1, 4\rangle, \langle 1, 7\rangle, \langle 1, 10\rangle, \ldots, \langle 2, 5\rangle, 2, 8, \ldots, \langle 0, 0\rangle, \langle 1, 1\rangle, \ldots \in R.$$

   Observe, that xRx for $x \in \mathbb{N}$ and, whenever xRy then also yRx.

3. Let $A = \mathbb{R}$, and define the relation R on $\mathbb{R}$ by xRy iff $y = x^2$. Then R consists of all points on the parabola $y = x^2$.

4. Let $A = \mathbb{R}$, and define R on $\mathbb{R}$ by xRy iff $x \cdot y = 1$. Then R consists of all pairs $\langle x, \frac{1}{x}\rangle$, where x is non-zero real number.

5. Let $A = \{1, 2, 3\}$, and define R on A by xRy iff $x + y = 7$. Since the sum of two elements of A is at most 6, we see that xRy for no two elements of A; hence, $R = \emptyset$.

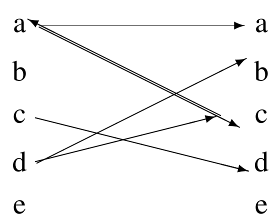
For small sets we can use a pictorial representation of a relation R on A: Sketch two copies of A and, if xRy then draw an arrow from the x in the left sketch to the y in the right sketch.

Let $A = \{a, b, c, d, e\}$, and consider the relation

$$R = \{\langle a, a\rangle, \langle a, c\rangle, \langle c, a\rangle, \langle d, b\rangle, \langle d, c\rangle\}. \tag{2.1}$$

An arrow representation of R is given in Fig. 2.1

Figure 2.1: Arrow representation



We observe that e does not appear at all in the elements of R, and that, for example, b is not the first component of any pair in R. In order to give names to the sets of those elements of A which are involved in R, we make the following

Definition 2.2.2. Let R be a relation on A. Then,

$$\operatorname{dom} R = \{x \in A : \text{ There exists some } y \in A \text{ such that } \langle x, y\rangle \in R\}.$$

$\operatorname{dom} R$ is called the *domain of R*.

$$\operatorname{ran} R = \{y \in A : \text{ There exists some } x \in A \text{ such that } \langle x, y\rangle \in R\}$$

is called the *range of R*.

Finally, $\operatorname{fld} R = \operatorname{dom} R \cup \operatorname{ran} R$ is called the *field of R*. Observe that $\operatorname{dom} R$, $\operatorname{ran} R$, and $\operatorname{fld} R$ are all subsets of A. $\square$

Example 2.2.2. 1. Let A and R be as in (2.1); then

$$\operatorname{dom} R = \{a, c, d\}, \ \operatorname{ran} R = \{a, b, c, d\}, \operatorname{fld} R = \{a, b, c, d\}.$$

2. Let $A = \mathbb{R}$, and define R by xRy iff $y = x^2$; then,

$$\operatorname{dom} R = \mathbb{R}, \ \operatorname{ran} R = \{y \in \mathbb{R} : y > 0\}, \ \operatorname{fld} R = R.$$

3. Let $A = \{1, 2, 3, 4, 5, 6\}$, and define R by xRy iff $x \lneq y$ and x divides y;

$$R = \{(1, 2), (1, 3), \ldots, (1, 6), (2, 4), (2, 6), (3, 6)\},$$

and

$$\operatorname{dom} R = \{1, 2, 3\},\ \operatorname{ran} R = \{2, 3, 4, 5, 6\},\ \operatorname{fld} R = A.$$

4. Let $A = \mathbb{R}$, and R be defined as $\langle x, y\rangle \in R$ iff $x^2 + y^2 = 1$. Then $\langle x, y\rangle \in R$ iff $\langle x, y\rangle$ is on the unit circle with centre at the origin. So, $\operatorname{dom} R = \operatorname{ran} R = \operatorname{fld} \mathbb{R} = \{z \in \mathbb{R} : -1 \lneq z \lneq 1\}$.

Definition 2.2.3. Let R be a relation on A; then $R^{\smile} = \{\langle y, x\rangle : \langle x, y\rangle \in R\}$ is called the *converse of R*. $\square$

We obtain the converse $R^{\smile}$ of R if we turn around all the ordered pairs of R; if we have a pictorial representation of R, this means that all existing arrows are reversed.

In our next definition we combine two relations to form a third one:

Definition 2.2.4. Let R and S be relations on A; then $R \circ S = \{\langle x, z\rangle :$ there is a $y \in A$ such that xRy and $ySz\}$. The operation $\circ$ is called the *composition* or the *relative product* of R and S. $\square$

Example 2.2.3. 1. Suppose that we have a pictorial representation of the relations R and S. The relation $R \circ S$ is the set of all pairs $\langle x, z\rangle$ such that x is in the left copy of A, z is in the right copy, and there is an arrow from x to z via an element in the centre copy of A.

2. Let $A = \mathbb{N}$ and R defined by xRy iff $x + 1 = y$, S defined by ySz iff $z = 2y$. Then $\langle x, z\rangle \in R \circ S$ iff $z = 2(x + 1)$:

$$\begin{aligned}\langle x, z\rangle \in R \circ S &\iff \text{There is some } y \in A \text{ with } xRySz\\ &\iff y = x + 1 \text{ and } z = 2y,\\ &\iff z = 2(x + 1).\end{aligned}$$

3. Let R be any relation on A; then

$$R \circ R^{\smile} = \{\langle x, z\rangle : x, z \in \operatorname{dom} R \text{ and there is some } y \in \operatorname{ran} R \text{ with } xRy \text{ and } zRy\} :$$

Note that on both sides of = we have a set, so, we have to show that two sets are equal.

Proof. "$\subseteq$": Let $\langle x, z\rangle \in R \circ R^{\smile}$; then there exists some $y \in A$ such that $xRyR^{\smile}z$, i.e. $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R^{\smile}$. Since $\langle x, y\rangle R$, we have $x \in \operatorname{dom} R$, and since $\langle y, z\rangle \in R^{\smile}$, we have $(z, y) \in R$; hence $z \in \operatorname{dom} R$. Furthermore, $y \in \operatorname{ran} R$, as well as xRy and zRy.

"$\supseteq$": Let $\langle x, y\rangle \in R$ and $(z, y) \in R$; then, $\langle y, z\rangle \in R^{\smile}$, and thus, $xRyR^{\smile}z$, i.e. $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R^{\smile}$. □

Observe that $\langle x, z\rangle \in R \circ R^{\smile}$ iff there are arrows from x and z which go to the same element y of A.

2.2.1 Exercises

Exercise 12. For the relation (2.1), find $R^{\smile}$, $R \circ R$, and $R^{\smile}\circ$R.

Exercise 13. The *identity* or *diagonal relation* on A is defined by

$$\varpi = \{\langle x, x\rangle : x \in A\}. \tag{2.2}$$

Show: For any relation R on A $R \circ \varpi = R$, and $\varpi \circ R = R$.

2.3 Ordering Relations

Thus far, we have not paid much attention to the structure a relation imposes on a set; in this section we shall define several *ordering relations*. As a first example, let $A = \mathbb{N}$ and let R be the relation defined by $\langle x, y\rangle \in R$ iff $x \leq y$; we note that $\leq$ (or R) has the properties that for all $x, y, z \in \mathbb{N}$,

1. $x \leq x$,
2. If $x \leq y$ and $y \leq x$ then $x = y$,
3. If $x \leq y$ and $y \leq z$ then $x \leq z$,
4. $x \leq y$ or $y \leq x$, i.e. any two elements of $\mathbb{N}$ are comparable with respect to $\leq$.

We shall use the first three properties to define our first type of ordering relation:

Definition 2.3.1. Let R be a relation on A.

1. R is *reflexive* if $\langle x, x\rangle \in R$ for all $x \in A$.

2. R is *antisymmetric* if for all $x, y \in A$, $\langle x, y\rangle \in R$ and $\langle y, x\rangle \in R$ implies $x = y$.

3. R is *transitive* if for all $x, y, z \in A$, $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$ implies $\langle x, z\rangle \in R$.

4. R is a *partial order* on A, if R is reflexive, antisymmetric, and transitive.

   Sometimes we will call a partial order on A just an *order on* A, or an *ordering of* A.

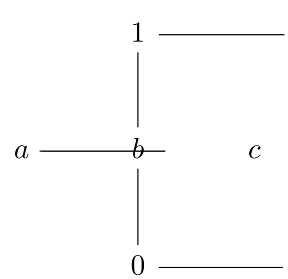
5. R is a *linear order on* A if R is a partial order, and xRy or yRx for all $x, y \in A$, i.e. if any two elements of A are comparable with respect to R.

If R is an ordering relation on A, then we usually write $\leq$ (or a similar symbol) for R, i.e. $x \leq y$ iff xRy. If $\leq$ is a partial order on A, then we call the pair $\langle A, \leq\rangle$ a *partially ordered set*, or just an *ordered set*. If $\leq$ furthermore is a linear order, then $\langle A, \leq\rangle$ is called a *linearly ordered set* or a *chain*. $\square$

For a finite partially ordered set $\langle A, \leq\rangle$ we can draw an *order diagram* by the following rules: If $a \leq b$ and $a \neq b$, then put b above a. b need not be directly above a, but also may be shifted to one side or the other. If there is no element between a and b, you connect them by a line.

Example 2.3.1. 1. Let $A = \{0, 1, a, b, c\}$, and define $\lneq$ by the diagram given in Fig. 2.2

Figure 2.2: The diamond



This diagram represents the following relation on A:

$$0 \leq 0,\ 0 \leq a,\ 0 \leq b,\ 0 \leq c,\ 0 \leq 1,\ a \leq a,\ a \leq 1,\ b \leq b,\ b \leq 1,\ c \leq c,\ c \leq 1,\ 1 \leq 1.$$

It is not hard to see that this is indeed a partial order on A.

2. Let $A = \{2, 3, 4, 5, 6\}$, and define R by the usual $\leq$ relation on $\mathbb{N}$, i.e. aRb iff $a \leq b$. Then R is a linear order on A.

3. Let us define another relation on $\mathbb{N}$ by

   (2.3) $$a/b \text{ iff } a \text{ divides } b.$$

   To show that / is a partial order we have to show the three defining properties of a partial order relation:

   Reflexive: Since every natural number is a divisor of itself, we have a/a for all $a \in A$.

   Antisymmetric: If a divides b then we have either $a = b$ or $a \lneq b$ in the usual ordering of $\mathbb{N}$; similarly, if b divides a, then $b = a$ or $b \lneq a$. Since $a \lneq b$ and $b \lneq a$ is not possible, a/b and b/a implies $a = b$.

   Transitive: If a divides b and b divides c then a also divides c.

   Thus, / is a partial order on $\mathbb{N}$.

4. Let $A = \{x, y\}$ and define $\leq$ on the power set $\mathfrak{P}(A)$ by $s \leq t$ iff s is a subset of t (see also 1.1 on page 11). This gives us the following relation:

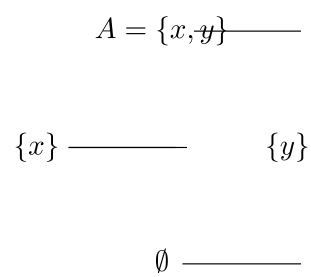
$$\emptyset \leq \emptyset,\ \emptyset \leq \{x\},\ \emptyset \leq \{y\},\ \emptyset \leq \{x,y\} = A,\ \{x\} \leq \{x\},\ \{x\} \leq \{x,y\}$$
$$\{y\} \leq \{y\},\ \{y\} \leq \{x,y\}\ \{x,y\} \leq \{x,y\}.$$

2.3.1 Exercises

Exercise 14. Let $A = \{1, 2, \ldots, 10\}$ and define the relation # on A by $x\#y$ iff x is a multiple of y. Show that # is a partial order on A and draw its diagram.

Exercise 15. What is the connection between $\#$ and $/$, defined in (2.3), on $A = \{0, 1, \ldots, 10\}$? What does the diagram of / look like?

Figure 2.3: 4 element powerset



2.4 Equivalence Relations

Let $\equiv$ be the relation on $\mathbb{N}$ defined by

$a \equiv b$ iff a and b have the same remainder when divided by 3.

Observe that every $x \in \mathbb{N}$ can be written either as $x = 3n$, or $x = 3n + 1$, or $x = 3n + 2$, depending on the remainder when x is divided by 3. Clearly, the relation $\equiv$ is reflexive; it is also transitive:

Let $a \equiv b$ and $b \equiv c$; then there are k, n, mN, and $r, s, t \lneq 2$ such that $a = 3k + r,\ b = 3n + s,\ c = 3m + t$. Since $a \equiv b$, we must have $r = s$, and since $b \equiv c$, we have $s = t$; thus, $r = s = t$, and we see that $a = 3k + r,\ c = 3m + r$, which tells us that $a \equiv c$. Hence, $\equiv$ is transitive.

However, $\equiv$ is not antisymmetric; take for example $a = 4$, and $b = 7$; then $4 \equiv 7$ and $7 \equiv 4$, since they both have remainder 1 when divided by 3. But this example indicates that $\equiv$ has another property: If $a \equiv b$ then $b \equiv a$, for any $a, b \in \mathbb{N}$. Many important relations have this property in addition to being reflexive and transitive, so they have been given a special name.

Definition 2.4.1. A relation R on a set A is called *symmetric* if $\langle a, b\rangle R$ implies $\langle b, a\rangle R$ for all a, bA; in other words, R is symmetric if $R = R^{\smile}$.

R is called an *equivalence relation* if R is reflexive, transitive and symmetric. $\square$

Example 2.4.1. The following are all equivalence relations:

1. Let A be the set of all squares and define the relation R by aRb if a and b have the same area.

2. Let A be the set of all straight lines in the plane and define R by gRh if $g = h$ or g and h are parallel.

3. Let A be the set of all cars in Iceland and define R by cRd if c and d have the same colour.

4. Let $A = \mathbb{N} \times \mathbb{N}$, i.e. the set of all pairs (n, m) where n and m are natural numbers. Define the relation R on A by $\langle n, m\rangle R\langle s, t\rangle$ if $n \cdot t = m \cdot s$. We will show that R is an equivalence relation, i.e. we prove that R has the three defining properties of an equivalence:

   (a) Show that R is reflexive, i.e. $\langle n, m\rangle R\langle n, m\rangle$ for all$n, m \in \mathbb{N}$.

   (b) R is transitive, i.e.

   $$\langle n, m\rangle R\langle p, q\rangle \text{ and } \langle p, q\rangle R\langle s, t\rangle \text{ imply that } \langle n, m\rangle R\langle s, t\rangle.$$

   (c) R is symmetric, i.e. if $\langle n, m\rangle R\langle s, t\rangle$ then $\langle s, t\rangle R\langle n, m\rangle$.

   We show only 4a, the rest will be left as an exercise. Let $\langle n, m\rangle \in A$. From our definition of R we know that $\langle n, m\rangle R\langle n, m\rangle$ iff $n \cdot m = n \cdot m$; this equation is true for all natural numbers, so we obtain $\langle n, m\rangle R\langle n, m\rangle$; hence R is reflexive.

The importance of equivalence relations lies in the fact that they induce a grouping of the base set into subsets. Let us look at two examples:

Example 2.4.2. 1. Let A be a set of objects each of which has only one colour; these could be blocks or balls or similar things. If we are teaching a child to distinguish colours, we might ask the child to make heaps of those objects which have the same colour. By doing this, we are in fact training the child to work with the relation R on A, which is defined by

$$xRy \text{ if } x \text{ and } y \text{ have the same colour.}$$

A heap of objects then would be the set $\{y \in A : xRy\}$ for some fixed $x \in A$; we would point to one object (our fixed x), say, a blue tile, and would ask the child to collect all blue objects (all y of the same colour). Note that each object of A is in exactly one pile.

2. At the beginning of the section we have looked at the relation R on $\mathbb{N}$, where xRy if x and y have the same remainder when divided by 3. This groups the natural numbers into three subsets:

$$A = \{0, 3, 6, 9, \ldots\},$$
$$B = \{1, 4, 7, 10, \ldots\},$$
$$C = \{2, 5, 8, 11, \ldots\}.$$

   Observe that $A = \{n \in \mathbb{N} : 0Rn\}$, $B = \{n \in \mathbb{N} : 1Rn\}$, $C = \{n \in \mathbb{N} : 2Rn\}$. Also, A, B, C are pairwise disjoint, and their union is all of $\mathbb{N}$.

Definition 2.4.2. Let A be a non-empty set. A family $\mathcal{P}$ of non-empty subsets of A is called a *partition of* A, if every element of A is in exactly one element of $\mathcal{P}$. In other words,

1. For all $S, T \in \mathcal{P}$ we have $S \cap T = \emptyset$,
2. The union of all elements of $\mathcal{P}$ is A.

The elements of the partition are called the *classes of* $\mathcal{P}$. A partition of A is also called a *classification*. □

Example 2.4.3. 1. The set of all students on a course can be partitioned in to the classes

   (a) $C1$ - the set of first year students.
   (b) $C2$ - the set of second year students.
   (c) $C3$ - the set of third year students.
   (d) $C4$ - the set of fourth year students.

2. Let A be a set of 23 men standing on a football field. We can partition A into two classes of eleven elements each (the teams), and one class with one element (the referee).

3. We can partition the set of all human beings at a specified time into classes according to the day and month of birth. This gives us 366 different classes.

If $\mathcal{P}$ is a partition of A, and if its classes are defined by certain properties, then each element of a specific class has the property which defines the class; in other words all the elements of a class are indistinguishable or equivalent with respect to the defining property of the class.

Our next aim is to show that equivalence relations and partitions are just two sides of the same coin: Each equivalence relation induces a unique partition and vice versa.

Definition 2.4.3. 1. Let R be an equivalence relation on A; for each $x \in A$, the set $\{y \in A : \text{xRy}\}$ is called the R – *class of* x, and it is denoted by Rx. The set $\{Rx : x \in A\}$ of all R – classes is denoted by $\mathcal{P}(R)$.

2. Let $\mathcal{P}$ be a partition of A; the relation $E(\mathcal{P})$ is defined on A as follows:

$$\langle x, y\rangle \in E(\mathcal{P}) \text{ if } x \text{ and } y \text{ are in the same class of } \mathcal{P}. \tag{2.4}$$

The main Theorem is the following:

Theorem 2.4.1. *1. Let R be an equivalence relation on* A*; then $\mathcal{P}(R)$ is a partition of A.*

2. Let $\mathcal{P}$ be a partition of A*; then $E(P)$ is an equivalence relation on A.*

3. $E(\mathcal{P}(R)) = R$ and $\mathcal{P}(E(\mathcal{P})) = \mathcal{P}$.

Proof. 1. By Definition 2.4.2 we have to show three things:

(a). Each R-class is not empty.

(b). If Rx and Ry are different R-classes, then their intersection is empty,

(c). The union of all R-classes is all of A, i.e. each element of A is in one R-class.

Since R is reflexive, we have for each $x \in A$ that xRx; this implies $x \in Rx$. Thus, every R-class is non – empty, and each element of $A$is in (at least) one R-class; this proves (a) and (c).

To prove (b), we first show the following:

Claim. If $y \in Rx$, then $Ry = Rx$.

Proof. Suppose that $y \in Rx$.

"$\subseteq$": Let $z \in Ry$. Then, by definition of Rx, resp. Ry, we have xRy and yRz. Since R is transitive, this implies xRz, and hence $z \in Rx$.

"$\supseteq$": For the converse, $z \in Rx$. Then, xRz, and we know from before that xRy as well. Since R is symmetric, we obtain that yRx. Again using transitivity we see that yRz, i.e. $z \in Ry$. This shows that $Rx \subseteq Ry$. □

Now, let Rx and Ry be different R-classes. Assume that $Rx \cap Ry \neq \emptyset$; then there exists some $z \in Rx \cap Ry$; since $z \in Rx$ we have xRz, and since $z \in Ry$ we have yRz. As R is symmetric, this implies that also zRy, giving us xRz and zRy. The transitivity of R then implies that xRy, i.e. x and y are in the same R-class. It follows from our claim that $Rx = Ry$, which contradicts our hypothesis that they be different. So, our assumption must be wrong, and therefore, $Rx \cap Ry = \emptyset$.

2. By definition of a partition, each element x of A is in exactly one class. Since each x is a member of its own class we have $xE(\mathcal{P})x$, and thus $E(P)$ is reflexive.

If x is in the same class as y, then obviously y is in the same class as x; thus, $xE(\mathcal{P})y$ implies $yE(\mathcal{P})x$, and $E(\mathcal{P})$ is symmetric.

Finally, let $xE(\mathcal{P})y$ and $yE(\mathcal{P})z$; then, x and y are in the same class, and y and z are in the same class. Since different classes have no common elements, x and z must be in the same class, i.e. $xE(\mathcal{P})$z. This shows that $E(\mathcal{P})$ is transitive.

3. For the first part, let $xE(\mathcal{P}(R))y$; then x and y are in the same class of $\mathcal{P}(R)$, i.e. $Rx = Ry$, which implies xRy.

Conversely, if xRy, then $Rx = Ry$ implying $xE(\mathcal{P}(R))y$.

The proof of the second part is similar and is left as an exercise. □

2.4.1 Exercises

Exercise 16. Find all partitions of $\{a, b, c, d\}$.

Exercise 17. Finish the proof of Theorem 2.4.1 on the previous page.

Exercise 18. Prove that for an equivalence relation R, $R = R^{\smile}$ and $R \circ R = R$.

Chapter 3

Functions

3.1 Basic Definitions

Definition 3.1.1. A *function* is an ordered triple $\langle f, A, B \rangle$ such that

1. A and B are sets, and $f \subseteq A \times B$,
2. For every $x \in A$ there is some $y \in B$ such that $\langle x, y \rangle \in f$
3. If $\langle x, y \rangle \in f$ and $\langle x, z \rangle \in f$, then $y = z$; in other words, the assignment is unique in the sense that an $x \in A$ is assigned at most one element of B.

A is called the *domain of* f, and B its *codomain*.

It is customary to write the function $\langle f, A, B \rangle$ as $f : A \to B$. Also, if $\langle x, y \rangle \in f$, then we will usually write $y = f(x)$, and call y the *image of* x *under* f.

The set $\{y \in B :$ There is an $x \in A$ such that $y = f(x)\}$ is called *the range of* f.
$\square$

Observe that the range of f is always a subset of the codomain. Observe carefully the distinction between $f(x)$ and f: Whereas $f(x)$ is an element of the codomain, f is the rule of assignment, conveniently expressed as a subset of $A \times B$.

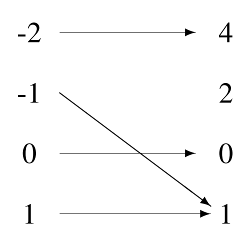
Suppose that $f : A \to B$ and $g : C \to D$ are functions. It follows from the definition of a function that they are equal if and only if

1. $A = C$,
2. $B = D$,
3. $f = g$.

If for a function $f : A \to B$ it is clear what A and B are, we sometimes call the function simply f, but we must keep in mind that a function is only properly defined if we also give a domain and a codomain!

Usually, after having agreed on a domain and a codomain, f is given by a rule, e.g. $f(x) = x^2$, $f(t) = \sin t, f(n) = n + 1$. As in the previous section with relations, we sometimes use a diagram to describe a function; for example, the diagram of Figure 3.1 describes the function $f : A \to B$, where $A = \{1, -1, 0, -2\}$ is the domain of $f, B = \{1, 0, 2, 4, \}$ is its codomain, and $f = \{\langle 1, 1\rangle, \langle -1, 1\rangle, \langle 0, 0\rangle, \langle -2, 4\rangle\}$.

Figure 3.1: Function arrow diagram



Observe that for all $x \in A$, $f(x) = x^2$. This can also be indicated by writing $x \mapsto x^2$. The definition of a function implies that each element of A is the origin of exactly one arrow; it does not imply that at each element of B is the target of an arrow, or that only one arrow from A points to a single element of B. Functions with these properties have special names, and we shall look at them in a later section.

Definition 3.1.2. Let $f : A \to B$ be a function.

1. If $A = B$ and $f(x) = x$ for all $x \in A$, the f is called the *identity function on* A, and it is denoted by id_A. Compare this with Exercise 13 on page 27.
2. If $A \subseteq B$ and $f(x) = x$ for all $x \in A$, then f is called the *inclusion function from* A *to* B, or, if no confusion can arise, simply the inclusion. Observe that, if $A = B$ and f is the inclusion, then f is in fact the identity on A.

3. If $f(x) = x$ for some $x \in A$, then x is called a *fixed point of f*.

4. If $f(x) = b$ for all $x \in A$, then f is called a *constant function*.

5. If $g : C \to D$ is a function such that $A \subseteq C,\ B \subseteq D$, and $f \subseteq g$, then g is called an *extension of f over C*, and f is called the *restriction of g to A*. □

While a function may have many different extension, it can only have one restriction to a subset of its domain. The following diagrams shall illustrate these situations: Consider the assignment $h(x) = (\sin x)^2$, and suppose that $\operatorname{dom} h = \operatorname{codom} h = \mathbb{R}$. To find $h(x)$ for a given x, we do two things:

1. First find $\sin x$ and set $y = \sin x$;

2. Then find y^2.

If we look close enough, we find that we have actually used two functions to find $h(x)$:

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = \sin x$,

2. $g : \mathbb{R} \to \mathbb{R},\ g(y) = y^2$,

This shows that $h(x) = g(f(x))$: We have first applied f to x, found that $f(x)$ is an element of $\operatorname{dom} g$, and then applied g to $f(x)$. This brings us to

Definition 3.1.3. Let $f : A \to B$ and $g : C \to D$ be functions such that $\operatorname{ran} f \subseteq \operatorname{dom} g$. Then the function $g \circ f : A \to C$ defined by $(g \circ f)(x) = g(f(x))$ is called the *(functional) composition* of f and g. □.

Note that this is different from the relational composition of Definition 2.2.4 on page 26; the reason for using a different interpretation of composition for functions is historical, and we shall not go into details.

Lemma 3.1.1. *If f and g are functions such that* $\operatorname{ran} f \subseteq \operatorname{dom} g$*, then* $\operatorname{dom}(g \circ f) = \operatorname{dom} f$*, and* $\operatorname{codom}(g \circ f) = \operatorname{codom} g$*.*

Proof. This follows immediately from the definition of composite functions. □

One the most useful properties of functional composition is the following:

Lemma 3.1.2. *Let* $f : A \to B,\ g : B \to C,\ h : C \to D$ *be functions. Then,* $h \circ (g \circ f) = (h \circ g) \circ f$*, i.e. the composition of functions is associative.*

Proof. Let $p = h \circ (g \circ f)$, and $q = (h \circ g) \circ f$; to show that two functions are equal we use the remarks following Definition 3.1.1 on page 35; thus, we first have to find their domains and co-domains.

Looking first at p, we see that p is the composite of the functions $g \circ f$ and h, so, we will have to look at $g \circ f$ first. Now, $\operatorname{dom}(g \circ f) = \operatorname{dom} f = A$ by Lemma 3.1.1, thus, $\operatorname{dom} p = \operatorname{dom} h \circ (g \circ f) = A$. Next,

$$\operatorname{codom} p = \operatorname{codom} h \circ (g \circ f) = \operatorname{codom} h = D,$$

also by Lemma 3.1.1.

Looking at q, we see by a similar reasoning that $\operatorname{dom} q = A$ and that $\operatorname{codom} q = \operatorname{codom}(h \circ g)$. Since $\operatorname{codom}(h \circ g) = \operatorname{codom} h$, we have $\operatorname{codom} h = D$; thus, we find that $\operatorname{dom} p = \operatorname{dom} q$, and $\operatorname{codom} p = \operatorname{codom} q$.

All that is left to show is that $p = q$, i.e. that $p(x) = q(x)$ for all $x \in A$. Let $x \in A$; then,

$$\begin{aligned}
p(x) &= (h \circ (g \circ f))(x) \\
&= h((g \circ f)(x)) \\
&= h(g(f(x))) \\
&= (h \circ g)(f(x)) \\
&= ((h \circ g) \circ f)(x) \\
&= q(x).
\end{aligned}$$

Thus, $(p, A, D) = (q, A, D)$. $\square$

3.1.1 Exercises

Exercise 19. For each of the following cases find, if possible, the domain, codomain and rule of assignment for $g \circ f$ and $f \circ g$:

1. $f : \mathbb{N} \to \mathbb{R},\ f(n) = n + 1$; $g : \mathbb{R} \to \mathbb{R},\ g(x) = \sqrt[3]{x}$.
2. $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = -x$; $g : \mathbb{R} \to \mathbb{R},\ g(x) = \sin x$.
3. $f : \mathbb{R} \to \mathbb{R},\ f(x) = 3x$; $g : \mathbb{R} \to \mathbb{R},\ g(x) = \frac{x}{3}$.

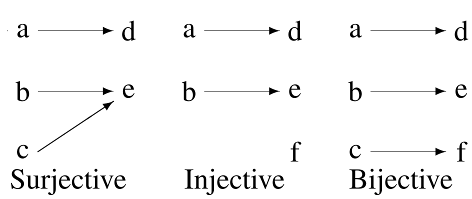
3.2 One–one, Onto, and Bijective Functions

Definition 3.2.1. Let $f : A \to B$ be a function.

1. f is called *onto* or *surjective* if $\operatorname{codom} f = \operatorname{ran} f$. If f is surjective, we sometimes indicate this by writing $f : A \twoheadrightarrow B$.
2. f is called *one–one* or *injective* if for all $x, y \in A$, $f(x) = f(y)$ implies $x = y$. If f is injective, we sometimes indicate this by writing $f : A \hookrightarrow B$.
3. f is called *bijective if* f is onto and one-one. □

Thus, f is onto if, in the pictorial representation, at least one arrow arrives at every element of the codomain; in other words, for every $y \in \operatorname{codom} f$ there exists at least one element x of $\operatorname{dom} f$ with $f(x) = y$. If f is one–one, then at most one arrow arrives at every element of $\operatorname{codom} f$. If f is bijective, then exactly one arrow arrives at every element of $\operatorname{codom} f$, see Figure 3.2.

Figure 3.2: Types of functions



Example 3.2.1. 1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = 2x$:

f is one–one: We have to show that for all $x, y \in \mathbb{R}$, $f(x) = f(y)$ implies $x = y$. So let $f(x) = f(y)$; by definition of f, $f(x) = 2x$ and $f(y) = 2y$. Since $f(x) = f(y)$, the definition of f tells us that $2x = 2y$, which implies $x = y$.

f is onto: For every real number y we have to find a real number x such that $f(x) = y$, which, by our definition of f is the same as $2x = y$. Solving this equation for x gives us $x = \frac{y}{2}$; so, $f(\frac{y}{2}) = y$.

Since f is one–one and onto, it is bijective.

2. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$:

f is not one–one: To show that an object does not have a certain property, it is enough to exhibit one counterexample, So, to show that f is not one–one we have to find two different real numbers x and y such that $f(x) = f(y)$. This is easily done by choosing $x = 1$ and $y = -1$.

f is not onto: Again, we have to find just one counterexample. If e.g. $y = -1$, then for no real number x we have $f(x) = y$, since the square of a real number is never negative.

3. Let $\mathbb{R}^+$ be the set of positive real numbers and define $f : \mathbb{R} \to \mathbb{R}^+$ by $f(x) = x^2$. Observe that we have only changed the codomain of the previous example. The function still is not one–one, but now it is onto, since every positive real number is a square.

4. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be defined by $f(x) = x^2$. We have only changed the domain of f from Example 2. This function is not onto – the codomain is still all of $\mathbb{R}$, and negative numbers still cannot be squares of real numbers. But our new f is one–one, since every real number has at most one positive root.

These examples tell us that the properties of being one–one or onto depend not only on the rule of assignment, but also on the domain and the codomain of the function. In fact, by changing the codomain of a function, we can always obtain a surjective function with the same domain and the same rule of assignment:

Lemma 3.2.1. *Let $f : A \to B$ be a function; then $g : A \to \operatorname{ran} f$ defined by $g(x) = f(x)$ for all $x \in A$ is onto.*

Proof. Since $\operatorname{dom} f = \operatorname{dom} g$, and $g(x) = f(x)$ for all $x \in A,$ g is a function. To show that g is onto, let $y \in \operatorname{codom} g$. By our definition of g, $\operatorname{codom} g = \operatorname{ran} f$; thus there is an $x \in A = \operatorname{dom} f$ such that $f(x) = y$. Now, since $\operatorname{dom} f = \operatorname{dom} g$ we have $x \in \operatorname{dom} g$; furthermore, since $f(x) = g(x)$, we obtain $g(x) = y$. This shows that g is onto. $\square$

Each function $f : A \to B$ can be written as a composition $f = h \circ g$ of functions, where g is surjective, and h is injective. This is an important Theorem which is used in many algebraic contexts. To prepare for a proof, we need

Definition 3.2.2. Let $f : A \to B$ be a function. Then the *canonical equivalence relation θ_f of f* is defined by

$$x\theta_f y \iff f(x) = f(y)$$

for all $x, y \in A$.

The set $\{\theta_f x : x \in A\}$ of equivalence classes of θ_f is called the *quotient set of* A *with respect to* f, written as A/θ_f, or simply *quotient set*, if f is understood. $\square$

Theorem 3.2.2. *Let* $f : A \to B$ *be a function. Then, there are a set* C*, and functions* $g : A \twoheadrightarrow C$*,* $h : C \hookrightarrow B$ *such that* g *is surjective,* h *is injective, and* $f = h \circ g$*.*

The situation is picture in Figure 3.3

Figure 3.3: Mapping Theorem

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, two heads, "g"'] & B \\
C \arrow[r, hook, "h"'] & {}
\end{tikzcd}$$

Proof. Set $C = A/\theta_f$, and let$g : A \to A/\theta_f$ be defined by $x \overset{g}{\mapsto} \theta_f x$. Since each element of A is contained in exactly one class of $E(\theta_f)$, h is a surjective function.

Now, let $h : A/\theta_f \to B$ be defined as follows: Let $t \in A/\theta_f$. Then, there is some $x \in A$, such that $t = \theta_f x$; set $h(t) = f(x)$. We need to show that the definition of h is independent of the choice of the representative x of the class t. Thus, let $y \in \theta_f x$. By definition of θ_f, we have $x\theta_f y$ if and only if $f(x) = f(y)$, and it follows that $f(y) = f(x)$ for all $y \in \theta_f x$.

Finally,

$$(g \circ h)(x) = g(\theta_f x) = f(x),$$

which proves our claim. $\square$

3.2.1 Exercises

Exercise 20. Which of the following functions are one–one, onto, or bijective? Justify your answer.

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = \sin x$.
2. $g : \mathbb{N} \to \mathbb{N},\ g(n) = n + 1$.
3. $h : \mathbb{N} \to \mathbb{N}^+,\ h(n) = n + 1$.
4. $r : \mathbb{R} \to \{0\},\ r(x) = 0$.

Exercise 21. Let $f : A \to B$ be a function. Show that θ_f is an equivalence relation on A.

3.3 Inverse Functions and Permutations

Recall Definition 2.2.3 on page 26 of the converse $R^{\smile}$ of a relation R on a set A; we have obtained the converse $R^{\smile}$ of R by turning around all the arrows, resp. the order of the components. This was always possible, since a relation was just a set of ordered pairs with no other conditions attached.

For a function, reversing the arrows need not result in another function: Consider the examples in Figure 3.2 on page 39. In the surjective function, turning around the arrows does not result in a function, because there are two arrows leaving e.

Furthermore, you will notice that applying f to an element of $\operatorname{dom} f$ and then applying g to the result gets us right back to where we started, in other words, $g(f(a)) = a$. This property is decisive, and leads to the following

Definition 3.3.1. Let $f : A \to B$ and $g : B \to A$ be functions; g is called the *inverse of* f, denoted by f^{-1}, if $g(f(x)) = x$ for all $x \in A$. □

In other words, g is an inverse of f if and only if $g \circ f = id_A$. The examples above suggest that a function which is not bijective cannot have an inverse. The following Theorem shows that this observation is correct.

Theorem 3.3.1. *$f : A \to B$ has an inverse if and only if f is injective.*

Proof. "$\Rightarrow$": Suppose that $g : B \to A$ is an inverse of f; we have to show that f is one–one. Let $x, y \in A$ and $f(x) = f(y)$; then, $g(f(x)) = g(f(y))$. Since g is an inverse of f, we have $g(f(x)) = x$ and $g(f(y)) = y$, which implies $x = y$.

"$\Leftarrow$": Suppose that $f : A \to B$ is one–one; for each $y \in B$ we distinguish two cases:

1. $y \in \operatorname{ran} f$: Then there is exactly one $x \in A$ such that $f(x) = y$, since f is one–one. Now set $g(y) = x$.
2. $x \notin \operatorname{ran} f$: Choose an arbitrary element y_x of A, and set $g(y) = y_x$. So, $\operatorname{dom} g = B,\ \operatorname{codom} g = A$, and, since, f is one–one, each element of B is assigned exactly one element of A; hence, $g : B \to A$ is a function.

Finally, let $x \in A$; by definition of g we have $g(f(x)) = x$, which shows that g is an inverse of f. □

Note that in case f is one–one but not onto it has more than one inverse function, since g is defined arbitrarily for every $x \in B$ which is not in the range of f. Those elements of the codomain which are not in the range of f are in a way immaterial to the assignment f; from previous examples we know that a one–one function can be made bijective by restricting its codomain to the range. If f is bijective then there is only one inverse function:

Theorem 3.3.2. *If $f : A \to B$ is bijective, then f has unique inverse $g : B \to A$; furthermore, f is an inverse to g.*

Proof. Since f is bijective, it is one–one, and thus it has an inverse $g : B \to A$. Suppose that $h : B \to A$ is also an inverse to f, i.e. we have $h(f(x)) = x$ for every $x \in A$. Since in particular f is onto, for every $y \in B$ there is an $x \in A$ such that $f(x) = y$; now, since g is an inverse to f, we have $g(y) = g(f(x)) = x$, and since h is also an inverse to f, we have $h(y) = h(f(x)) = x$. It follows that $g(y) = h(y)$ for all $y \in B$, which shows that $g = h$.

For the second part we have to show that $f(g(y)) = y$ for all y B. Thus, let $y \in B$. Since f is onto, there exists an $x \in A$ such that $f(x) = y$. Thus,

$$f(g(y)) = f(g(f(x))) = f(x),$$

since g is an inverse of f, and hence, $g(f(x)) = x$. □

Let us briefly look at bijective functions $f : A \to A$, where A is a finite set.

Definition 3.3.2. Let $A = \{1, 2, 3, \ldots, n\}$, and $f : A \to A$ be a bijective function; then f is called a *permutation on n*. □

The reason for calling these function permutations is that they arrange (i.e. permute) the elements of Ain a different order. Amore general definition of a permutation would allow as domain arbitrary finite sets. For example, if you have n books arranged on a shelf and you put them in a different order, you have in fact performed a permutation of n objects.

If n is small there is a convenient way of picturing f; for example if $A = \{1, 2, 3, 4, 5\}$, and $f(1) = 1,\ f(2) = 3,\ f(3) = 5,\ f(4) = 2,\ f(5) = 4$, then we can write

k	1	2	3	4	5
$f(k)$	1	3	5	2	4

Observe that every element of A appears exactly once in each of the two lines, since f is bijective. In general, if $A = \{1, 2, \ldots, n\}$, and $f(1) = a_1, f(2) = a_2, \ldots, f(n) = a_n$, we can list the function by

k	1	2	3	…	n
$f(k)$	a_1	a_2	a_3	…	a_n

It is easy to find the inverse of a permutation by just looking from bottom to top in the given list. Let us go back to the example above: The inverse g of f looks like this:

k	1	2	3	4	5
$g(k)$	1	4	2	5	3

This method of representing a permutation is also useful for obtaining the composition; look at the following two permutations of A:

k	1	2	3	4	5
$f(k)$	2	3	4	1	5
$g(k)$	3	4	1	5	2

By going through the tables, we find that for $h = g \circ f$,

k	1	2	3	4	5
$h(k)$	4	1	5	3	2

Appendix A

Solutions to Exercises

Solution 1.

$$
\begin{aligned}
A &= \{\ldots, -2, -1, 0, 1, 2, 3, 4\}.\\
B &= \{1, 2, 3, 4, 6, 8, 12, 24\}.\\
C &= \{2\}.\\
D &= \emptyset.
\end{aligned}
$$

Solution 2.

$$
\begin{aligned}
A &= \{x^2 : x \in \mathbb{N}, 1 \leq x \leq 4\}.\\
B &= \{2x : x \in \mathbb{N}, 0 \leq x \leq 4\}.\\
C &= \{2x + 1 : x \in \mathbb{N}\}.\\
D &= \{x \in \mathbb{N} : 1 \leq x \leq 15 \text{ and } x \text{ is prime}\}.
\end{aligned}
$$

These descriptions are not unique; here are other possibilities:

$$
\begin{aligned}
A &= \{x^2 : x \in \mathbb{Z}, 1 \leq x \leq 4\}.\\
B &= \{\frac{x}{2} : x \in \{0, 4, 8, 16, 32\}\}.\\
C &= \{2x - 1 : x \in \mathbb{N}^+\}.\\
D &= \{x \in \mathbb{N} : 2 \leq x \leq 13 \text{ and } x \text{ is prime}\}.
\end{aligned}
$$

Solution 3. The powerset is

$$
\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\},\\
\{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\},
$$

and it has 16 elements.

Solution 4. We note that

- $\mathfrak{P}(\emptyset)$ has 1 element,
- $\mathfrak{P}(\{a\})$ has 2 elements,
- $\mathfrak{P}(\{a, b\})$ has 4 elements,
- $\mathfrak{P}(\{a, b, c\})$ has 8 elements,
- $\mathfrak{P}(\{a, b, c, d\})$ has 16 elements.

Thus, we conjecture that the powerset of a set with n elements has 2^n elements.

Solution 5. Suppose that $A = B$ and $B = C$. We must show that $A \subseteq C$ and $C \subseteq A$.

$A \subseteq C$: Let $x \in A$; since $A = B$, we have $x \in B$, and from $B = C$ it follows that $x \in C$.

$C \subseteq A$: Let $x \in C$; since $C = B$, we have $x \in B$, and from $B = A$ it follows that $x \in A$.

Solution 6. 1. There is no largest n such that $\mathbb{N}$ is numbered by $1, 2, \ldots n$, since there is always a natural number greater than any of a finite list, for example, the sum of all numbers in the list with 1 added.

2. Each set containing an infinite set as a subset must necessarily be infinite.

Solution 7. 1. Finite, since a quadratic equation in $\mathbb{R}$ has at most two solutions.

2. Finite, since it is empty.

3. Infinite.

Solution 8. 1. The second De Morgan rule is

$$-(A \cap B) = -A \cup -B.$$

A	B	$-A$	$-B$	$A \cap B$	$-(A \cap B)$	$-A \cup -B$
1	1	0	0	1	0	0
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	0	0	1	1

2. The second law of absorption is

$$A \cap (A \cup B) = A.$$

A	B	$A \cup B$	$A \cap (A \cup B)$
1	1	1	1
1	0	1	1
0	1	1	0
0	0	0	0

3. The first distributive law is

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Solution 9. $A \times B = \{\langle 2,4\rangle, \langle 2,8\rangle, \langle 2,12\rangle, \langle 4,4\rangle, \langle 4,8\rangle, \langle 4,12\rangle, \langle 6,4\rangle, \langle 6,8\rangle, \langle 6,12\rangle\}$.
$B \times A = \{\langle 4,2\rangle, \langle 4,4\rangle, \langle 4,6\rangle, \langle 8,2\rangle, \langle 8,4\rangle, \langle 8,6\rangle, \langle 12,2\rangle, \langle 12,4\rangle, \langle 12,6\rangle\}$.

Solution 10. Exactly if $A = B$.

Solution 11. For each element of A there are exactly m elements of B with which it can be paired. Thus, $A \times B$ has $n \cdot m$ elements.

Solution 12.

$$\begin{aligned} R^{\smile} &= \{\langle a,a\rangle, \langle c,a\rangle, \langle a,c\rangle, \langle b,d\rangle, \langle c,d\rangle\} \\ R \circ R &= \{\langle a,c\rangle, \langle a,a\rangle, \langle c,a\rangle, \langle c,c\rangle, \langle d,a\rangle\} \\ R^{\smile} \circ R &= \{\langle a,a\rangle, \langle a,c\rangle, \langle c,a\rangle, \langle c,c\rangle, \langle b,b\rangle, \langle b,c\rangle, \langle c,b\rangle\} \end{aligned}$$

Solution 13. We only show $R \circ \varpi = R$; the proof of the second part is analogous.

"$\subseteq$": If $\langle x,y\rangle \in R \circ \varpi$, then there is some z such that $xRz\varpi y$. Since ϖ is the identity, we must have $z = y$, and thus, xRy.

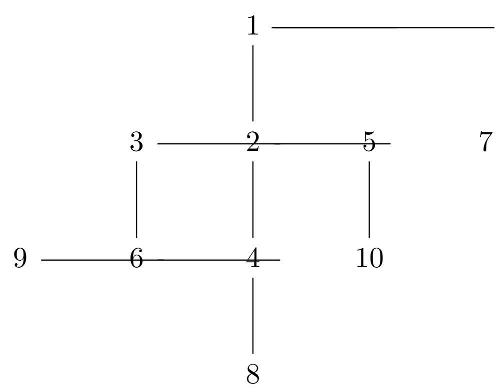
"$\supseteq$": If xRy, then $xRy\varpi y$, and therefore, $\langle x,y\rangle \in R \circ \varpi$.

Solution 14. Reflexive: Each natural number is a multiple of itself and 1. Since $1 \in A$, $x\#x$ for all $x \in A$.

Antisymmetric: If $x\#y$ and $y\#x$, then, in particular, $x \geq y$ and $y \geq x$, since the elements of A are natural numbers. Since $\geq$ is antisymmetric, we have $x = y$.

Transitive: Let $x, y, z \in A$ and $x\#y\#z$. Then, there are $t, u \in \mathbb{N}$ with $x = t \cdot y$ and $y = u \cdot z$. It follows that $x = t \cdot u \cdot z$.

Figure A.1: The ordering of Exercise 14



Solution 15. These relations are converses of each other, and the diagram of / is the diagram of # turned upside down.

Solution 16. One class:
$\mathcal{P}_1^1 = \{\{a, b, c, d\}\}$.

Two classes:
$\mathcal{P}_1^2 = \{\{a\}, \{b, c, d\}\}$, $\mathcal{P}_2^2 = \{\{b\}, \{a, c, d\}\}$, $\mathcal{P}_3^2 = \{\{c\}, \{a, b, d\}\}$, $\mathcal{P}_4^2 = \{\{d\}, \{a, b, c\}\}$, $\mathcal{P}_5^2 = \{\{a, b\}, \{c, d\}\}$, $\mathcal{P}_6^2 = \{\{a, c\}, \{b, d\}\}$, $\mathcal{P}_7^2 = \{\{a, d\}, \{b, c\}\}$.

Three classes:
$\mathcal{P}_1^3 = \{\{a\}, \{b\}, \{c, d\}\}$, $\mathcal{P}_2^3 = \{\{a\}, \{c\}, \{b, d\}\}$, $\mathcal{P}_3^3 = \{\{a\}, \{d\}, \{b, c\}\}$,
$\mathcal{P}_4^3 = \{\{b\}, \{c\}, \{a, d\}\}$, $\mathcal{P}_5^3 = \{\{b\}, \{d\}, \{a, c\}\}$, $\mathcal{P}_6^3 = \{\{c\}, \{d\}, \{a, b\}\}$,

Four classes:
$\mathcal{P}_1^4 = \{\{a\}, \{b\}, \{c\}, \{d\}\}$.

Solution 17.

$$\begin{aligned} M \in \mathcal{P}(E(\mathcal{P})) &\iff xE(\mathcal{P})y \text{ for all } x, y \in M \\ &\iff x \text{ and } y \text{ are in the same class of } \mathcal{P} \\ &\iff M \text{ is a class of } \mathcal{P}. \end{aligned}$$

Solution 18. 1. Since R is symmetric, we have $R = R^{\smile}$.

2. Transitivity of R implies that $R \circ R \subseteq R$. Conversely, let xRy; since R is reflexive, we have yRy, and it follows that $xRyRy$, i.e. $xR \circ Ry$.

Solution 19. 1. $\operatorname{dom} g \circ f = \mathbb{N}$, $\operatorname{codom} g \circ f = \mathbb{R}$, $n \mapsto \sqrt[3]{n+1}$.
Since $\operatorname{ran} g \not\subseteq \operatorname{dom} f$, we see that $f \circ g$ is not defined.

2. $\operatorname{dom} g \circ f = \mathbb{Z}$, $\operatorname{codom} g \circ f = \mathbb{R}$, $n \mapsto \sin(-x)$.
Since $\operatorname{ran} g \not\subseteq \operatorname{dom} f$, we see that $f \circ g$ is not defined.

3. $\operatorname{dom} g \circ f = \mathbb{R}$, $\operatorname{codom} g \circ f = \mathbb{R}$, $x \mapsto x$.
$\operatorname{dom} f \circ g = \mathbb{R}$, $\operatorname{codom} f \circ g = \mathbb{R}$, $x \mapsto x$.

Solution 20. 1. f is not injective, since e.g. $f(0) = f(2\pi)$, and not surjective, since $-1 \leq f(x) \leq x$ for all $x \in \mathbb{R}$.

2. g is injective since

$$g(n) = g(m) \Rightarrow n + 1 = m + 1 \Rightarrow n = m.$$

It is not surjective, since $n + 1 \neq 0$ for all $n \in \mathbb{N}$.

3. h is bijective, since $h(n) = g(n)$, and each $m \in \mathbb{N}^+$ is of the form $n + 1$ for some $n \in \mathbb{N}$ - set $n = m - 1$.

4. r is not injective, but surjective.

Solution 21. 1. Since $f(x) = f(x)$ for all $x \in A$, θ_f is reflexive.
2. Since $f(x) = f(y) \iff f(y) = f(x)$ for all $x, y \in A$, θ_f is symmetric.
3. If $x\theta_f y$ and $y\theta_f z$, then $f(x) = f(y)$ and $f(y) = f(z)$. Hence, $f(x) = f(z)$, and therefore $x\theta_f z$.

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